MATHEMATICAL MODELS

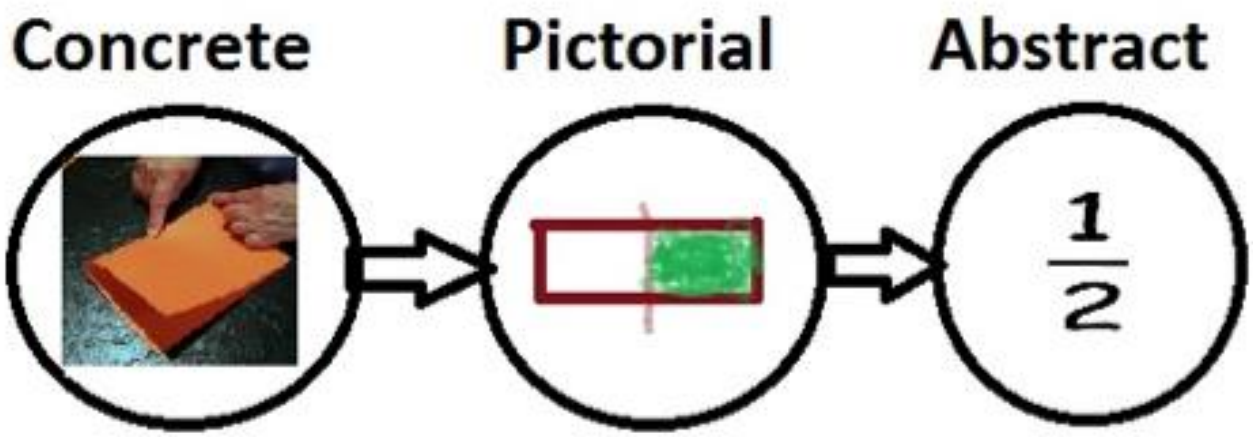
THE INTEGRATED DAY CHARTER SCHOOL

Math models provide a transition from abstract to concrete levels of processing, giving word problems and computation problems a visual context. These models are like graphic organizers that show the relationships among numbers. Students needs to learn how to use the model, automatize the process, and apply the process in new situations.

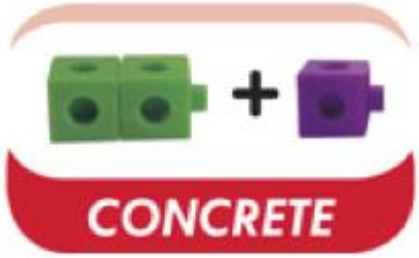
A Guide to Mathematical Modeling in Grades K-8

For most teachers, mathematical modeling represents a new way of "doing" mathematics that makes the addition of modeling activities into instruction seem daunting. This is especially true since modeling, when done properly, requires significant time and effort. In turn, some may be reluctant to include modeling activities into classroom time. It is essential to keep in mind that modeling is one of the eight Standards for Mathematical Practice given in the *Common Core State Standards for Mathematics* (CCSSM) for all grades and is a required conceptual category in high school. Because of this, ***modeling cannot be set aside or employed only when spare time arises***. Class time that previously may have been spent using more traditional teaching methods should be converted to time spent on modeling. The integrated nature of mathematical modeling, and in turn the number of curricular standards covered when working through a modeling activity, make modeling activities a very efficient use of class time.

--from *Mathematical Modeling Handbook*; Teachers College; Columbia University

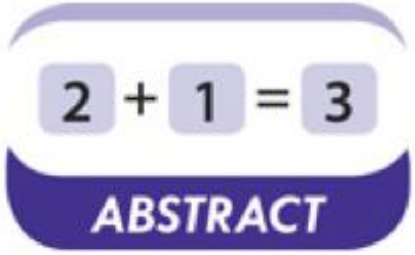


Concrete-Pictorial-Abstract Learning Progression – A Classical Approach to Laying a Strong Foundation

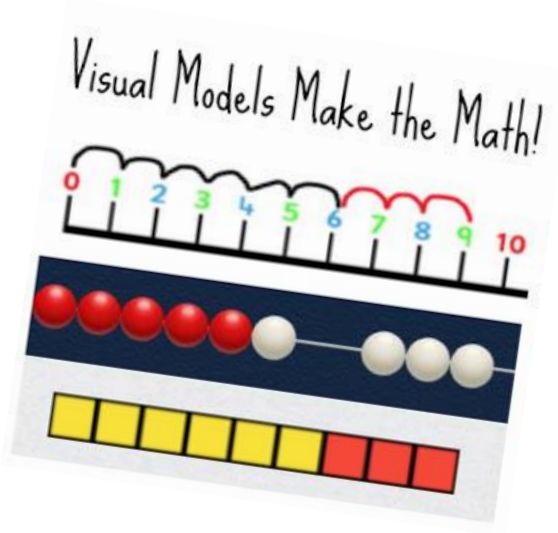


Concepts are introduced through hands-on experiences with manipulatives, the necessary first step for young children who think in concrete terms. Even older children may revert to this mode when learning a new skill.

Students visualize the concept and represent it pictorially through models like number bonds and bar models, which represents the "mental image" mode. This mode of thinking assists students in dissecting challenging word problems and developing logical thinking.

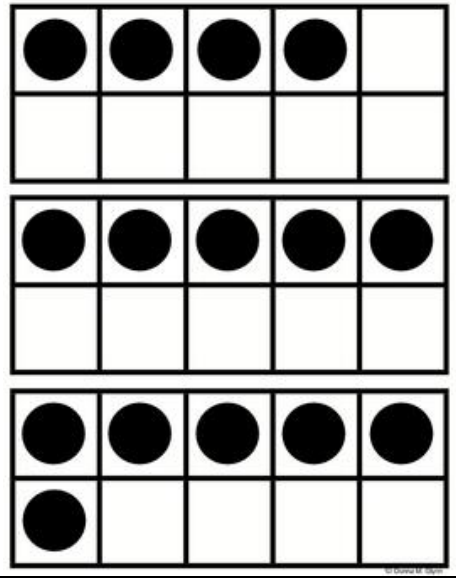


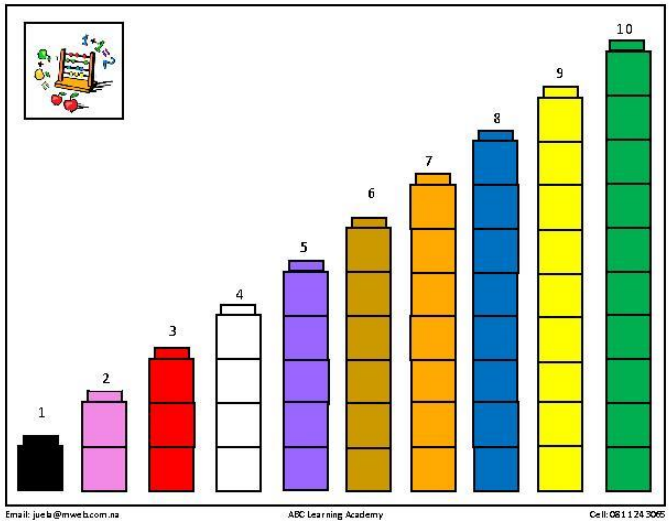
Students only use abstract numbers and symbols when they have enough context to understand what they mean, i.e. true abstract thinking without using concrete objects or pictures, a skill required by algebra, trigonometry and calculus later on.

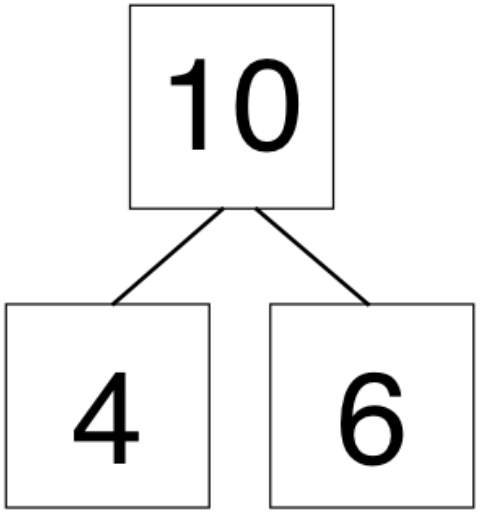
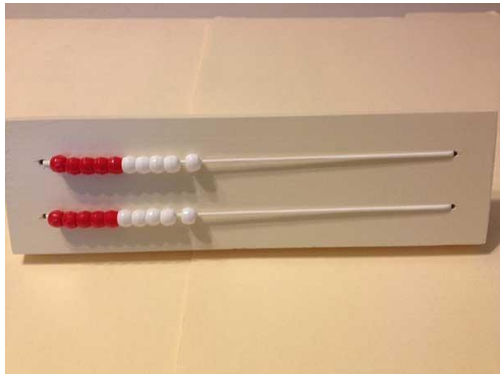


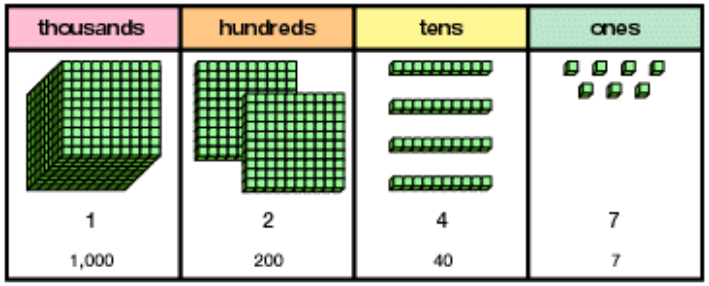
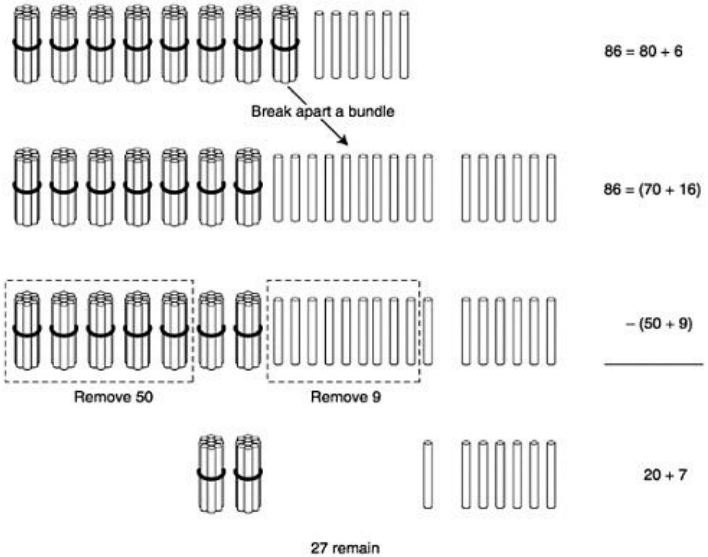
Teaching Tips for Using Mathematical Models:

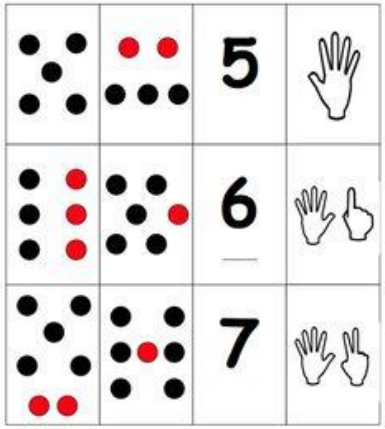
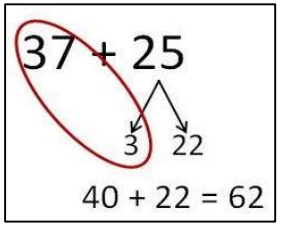
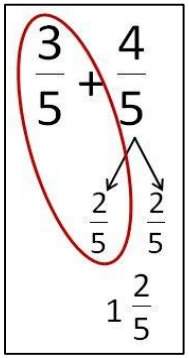
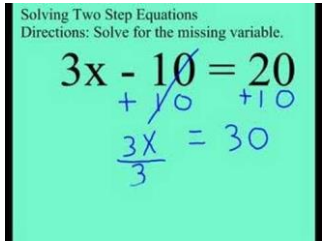
- "Reflectively" read the problems (without reading the numerals) before reading the entire problem
- Read the punctuation marks in the problems
- Have students place a checkmark over each numeral as they add it to the model
- Have students put a slash after each sentence to show they have finished adding the information to the model
- "Milk the problem for all it's worth" – additional questions using the same model
- Use graph paper or a vertical "starting line" to help students with alignment of unit bars
- Use individual objects, manipulatives, or illustrations to represent numerals in "number-bond" stories before introducing continuous model drawing

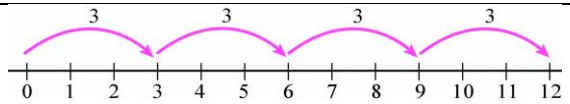
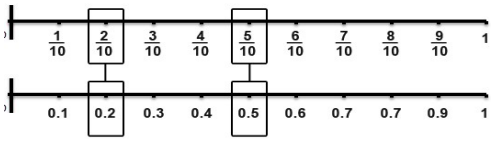
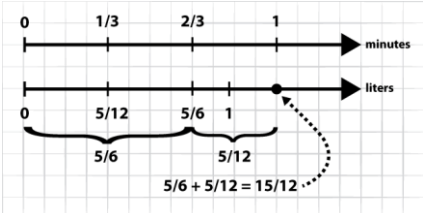
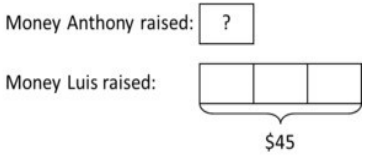
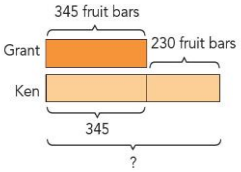
Term	Description	Visual Representation
Number Path	**Primary Grade Level: PK-1** The number path can be thought of as a visual (pictorial) representation of the number tower and is foundational to understanding and using the number line. It also serves as a visual representation of 1:1 correspondence and the concept of whole numbers (one number, one space, and each being equal in size). The color change at 5 helps to reinforce the 5 and 10 benchmarks. The number path also serves as an early precursor to measurement concepts and a support for cardinal counting.	
Ten Frame	**Primary Grade Level: PK - 3** A ten-frame is a 2 by 5 grid (array) used to develop an understanding of concepts such as 5-patterns, combinations to 10, and adding and subtracting within 20. The frame is filled beginning on the top row, left to right, then proceeding to the bottom row building left to right. This pattern of filling supports subitizing by building on the 5 benchmark, as well as providing a pattern for placing disks on place value mats in later grades. Concrete counters as well as pictorial dots may be used to represent quantities on the frame.	

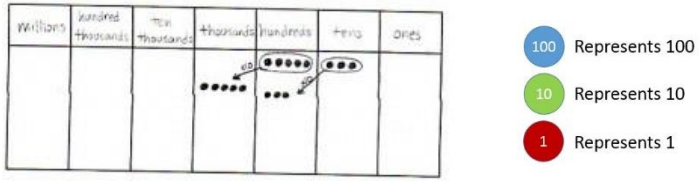
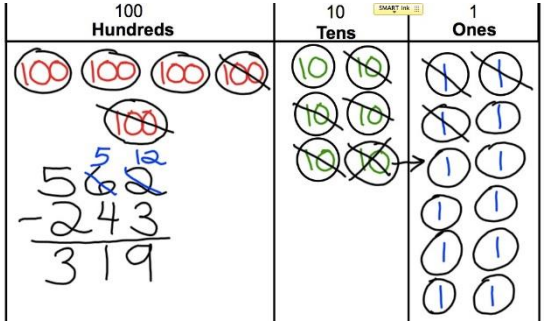
Number Towers	Primary Grade Level: PK-3 Number towers, also known as number stairs, are representations of quantity constructed by joining together interlocking cubes such as Unifix cubes. In the beginning, they are used to help younger students build their knowledge of cardinality by erecting towers of various numbers. Number towers are then used to teach concepts of “more/less” globally and the patterns of “1 more/less” and “2 more/less” specifically. This model leads to an understanding of comparison and the word “than,” not only in the context of “more than” and “less than,” but also in the context of “taller than,” “shorter than,” “heavier than,” “longer than,” etc. Children are encouraged to build towers for quantities 1 through 5 in one color. Quantities beyond 5 are added on in a second color. This color change provides support for understanding 5 as a benchmark, which is important to their ability to subitize. It also encourages students to count on from 5 rather than starting at 1 to count. In grades 2 and 3, each unit in a number stair can be ascribed a value other than 1. For example: “Each of our cubes is equal to three. What is the value of the stair with five cubes?”	 

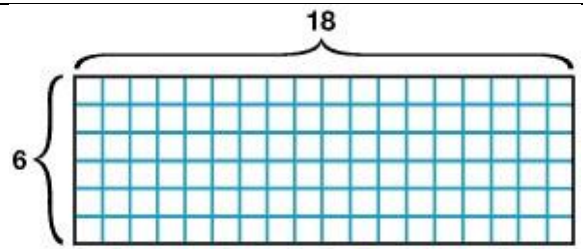
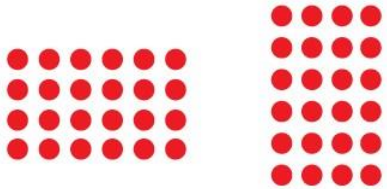
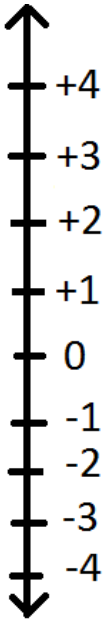
Number Bonds	Primary Grades K-5 The number bond is a pictorial representation of part-part-whole relationships and shows that within a part-whole relationship, smaller numbers (the parts) make up larger numbers (the whole). Number bonds of 10 have the greatest priority because students will use them for adding and subtracting across ten. Please note that the orientation of the number bond does not change its meaning and function. In the upper grades, numbers bonds can be used to see part-whole fraction and decimal relationships.	
Rekenrek	Primary Grade Level: PK-5 The Rekenrek has a 5 and 10 structure, with a color change at 5 (eliciting the visual effect of grouping 5 and 10). The 20-bead Rekenrek consists of 2 rows of 10 beads, allowing students to see numbers to 10 either as a number line on one row or a ten-frame (5 beads on two rows). A 100-bead Rekenrek has 10 rows of 10 beads. Other names for the Rekenrek are "Calculating Frame," "Slavonic Abacus," "Arithmetic Rack," or "Math Rack." In **grades PK-1**, use for skip-counting, making ten, add and subtracting within 10, compose and decompose numbers. In **grades 2-5**, use to model rectangular arrays to build conceptual understanding of multiplication, demonstrate the distributive property. Think of 3 x 12 as 3 x 10 plus 3 x 2.	 

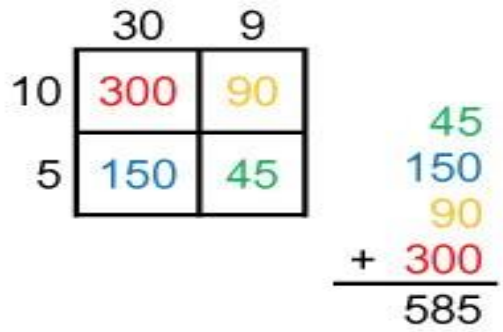
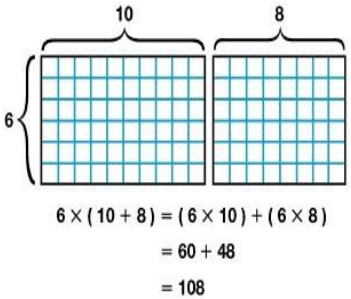
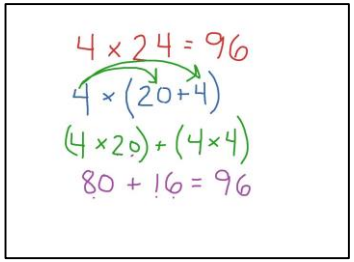
Base Ten Blocks	Primary Grade Level K-2 Include thousands “cubes,” hundreds “flats,” tens “rods,” and ones “units.” Base-ten blocks are a proportional representational of units and are useful for developing place value understanding. This is a “pre-grouped” model for base-ten that allows for more efficient modeling of larger quantities through the thousands. However, because this place value model requires students to more abstractly consider the 10 to 1 relationship of the various blocks, care must be taken to ensure that students attend to the “ten-ness” of the pieces that are now traded rather than bundled or un-bundled. Use for modeling addition, subtraction, multiplication, and division. Use blocks and mats as a support for teaching students to record the standard algorithms for all four operations.	
Bundle	Primary Grade Level: K-2 Bundles are discrete groupings of place value units (tens, hundreds, thousands), usually made my students/teachers placing a rubber band around straws or popsicle sticks. Linking cubes may also be used. Ten straws (or cubes) are bundled (or linked) into one unit of ten, 10 tens are bundled into one unit of a hundred, and so on. These student-made groupings provide the necessary conceptual foundation for children to be successful with pre-grouped, proportional, and non-proportional base-ten materials. Bundling and unbundling are central ideas in developing computation algorithms.	

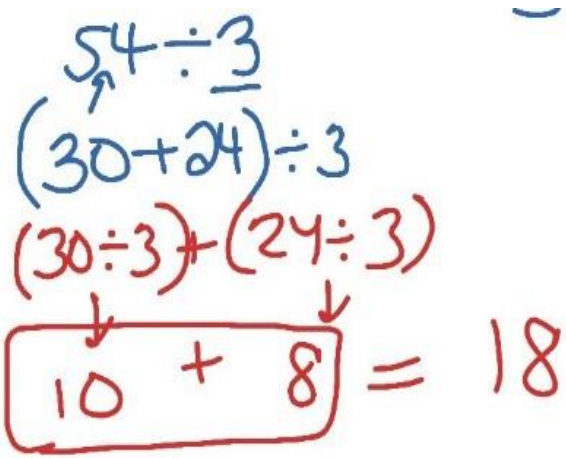
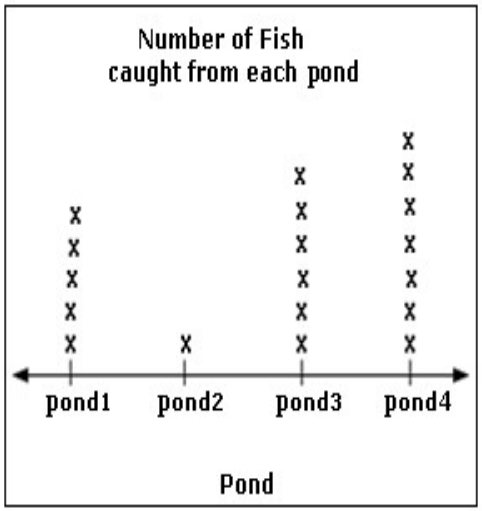
Subitize	Primary Grades: K-3 Subitizing is the ability to 'see' a small amount of objects and know how many there are without counting. Subitizing is what tells you what number you roll on a six sided dice – without counting each dot. Subitizing is a fundamental skill in the development of students' understanding of number. Students use patterns recognized to discover properties and skills such as conservation, compensation, unitizing, counting on, composing and decomposing numbers, as well as understanding of arithmetic and place value.	
Decompose	Primary Grades: K-8 Breaking numbers down into smaller parts to make them easier to manipulate/calculate.	 
Equation	Primary Grades: K – 8 A number sentence consisting of two values that are equal	64 + 23 = 87 2 x 5 = 9 + 1 

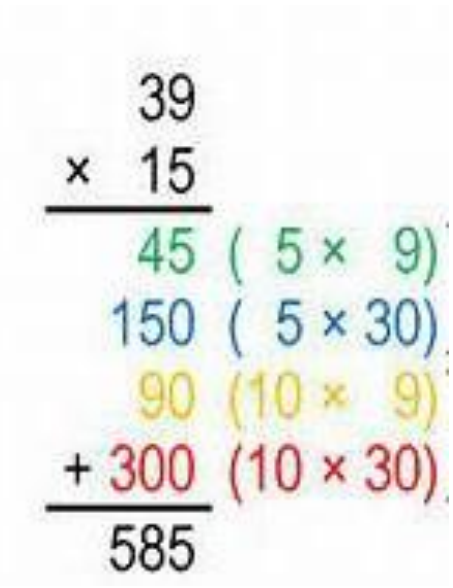
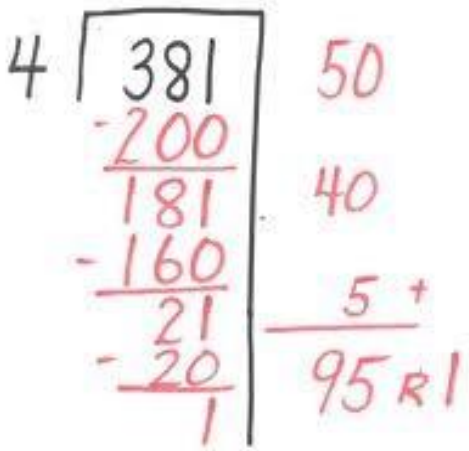
Number Line	Primary Grade Level: K-8 Used to develop deeper understanding of whole number units, fraction units, measurement units, decimals, and negative numbers. Throughout grades K-5, the number line models measuring units. Can be used for solving operations (addition, subtraction, multiplication, and division), measure lengths, rounding, model tenths in unit, expanded, fraction, and decimal form, ratios, and create and analyze line plots.	 4 x 3 = 12  
Tape Diagram (bar model)	Primary Grade Level: 1-8 Provides an essential bridge to algebra – is often called "pictorial algebra." Pictorial representations of relationships between quantities used to solve word problems Students begin using tape diagrams in 1st grade, modeling simple word problems involving the four operations. It is common for students in 3rd grade to express that they don't need the tape diagram to solve the problem. However, in grades 4 and 5, students begin to appreciate the tape diagram as it enables students to solve increasingly more complex problems.	Luis raised 3 times as much money as Anthony did. If Luis raised $45, how much money did Anthony raise?  Ken has 230 more fruit bars than Grant does. If Grant has 345 fruit bars, how many fruit bars does Ken have? 

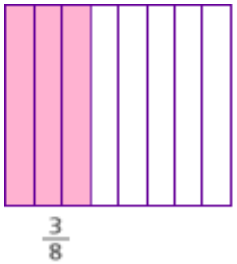
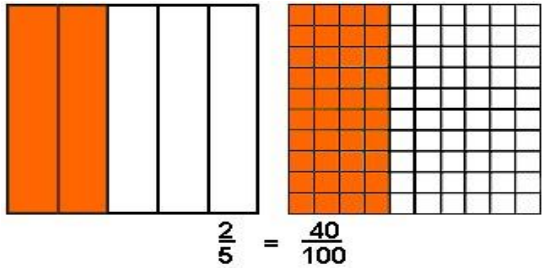
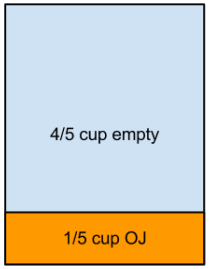
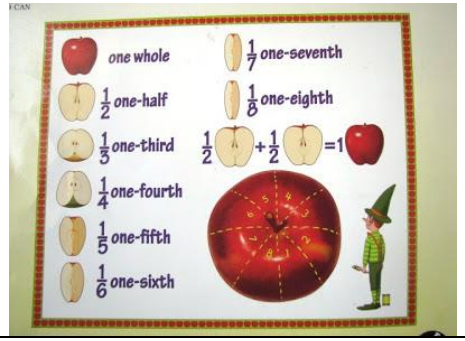
Money	Primary Grade Level: 2-3 Dollar bills (1s, 10s, and 100s) are non-proportional units that are used to develop place value understanding. That is, bills are an abstract representation of place value because their value is not proportionate to their size. Ten bills can have a value of $10 or $1000 but appear identical aside from their printed labels. Bills can be “traded” (e.g. 10 ten-dollar bills for 1 one-hundred-dollar bill) to help students learn equivalence of the two amounts. The transition from a discrete unit of a “bundle” to proportional materials such as base-ten blocks to a non-proportional unit of a bill is significant leap in a student’s place value learning trajectory.	
Place Value Disks	Primary Grades: 2-5 Number disks are non-proportional units used to further develop place value understanding. Like money, the value of the disk is determined by the value printed on it, not by its size. Number disks are used by students through Grade 5 when modeling algorithms and as a support for mental math with very large whole numbers. Whole number place value relationships modeled with the disks are easily generalized to decimal numbers and operations with decimals. Number Disks are used to perform all four operations with both while numbers and decimals on math. Use materials to bridge to recording the standard algorithm for all four operations with both whole numbers and decimals.	 

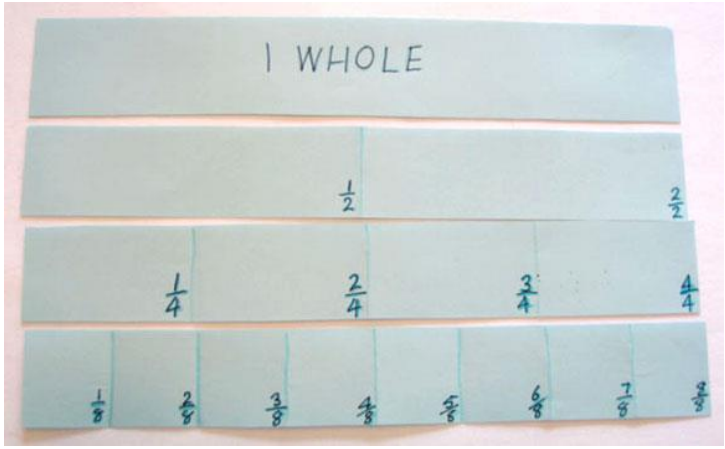
Array	Primary Grades: 2-5 Objects organized into equal rows (going horizontally) and columns (going vertically) Used for multiplication	 6 × 18 = 108  4 x 6 = 24 6 x 4 = 24
Vertical Number Line Model	Primary Grades 2 – 8 A vertical number line is one that goes straight up and down, parallel to the y-axis of the coordinate plane. All points on the line will have the same x-coordinate. Used for rounding and working with negative numbers	

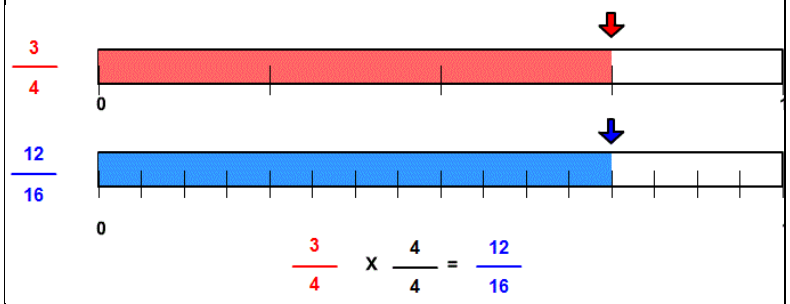
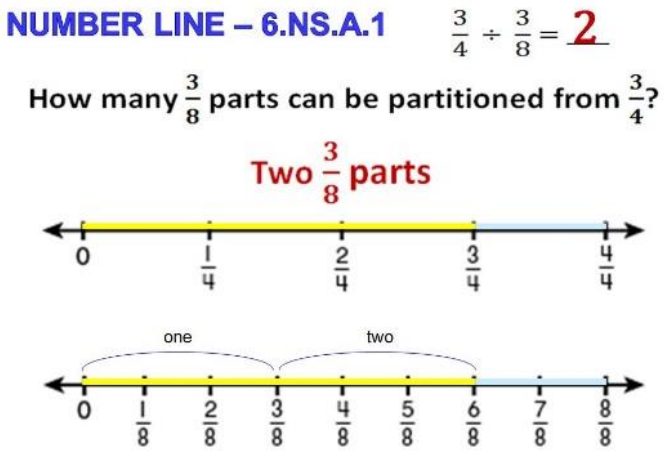
Area Model for Multiplication	Primary Grades: 3-5 Like an array without individual boxes Used for multiplying The digits are broken up into their values and multiplied – products are then added together	
Distributive Property (Multiplication)	Primary Grades: 3-8 States that multiplying a sum by a number is the same as multiplying each addend by the number and then adding the products	6 x 18 = 108  

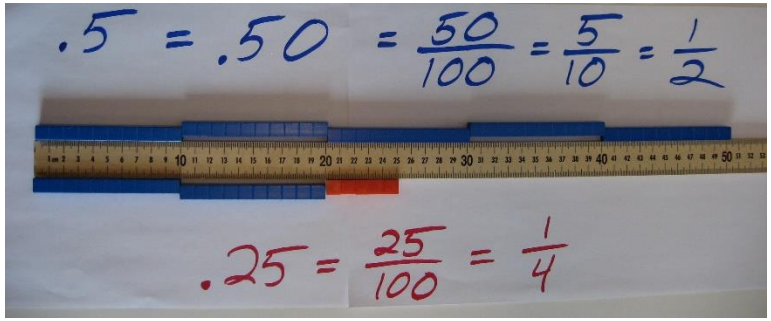
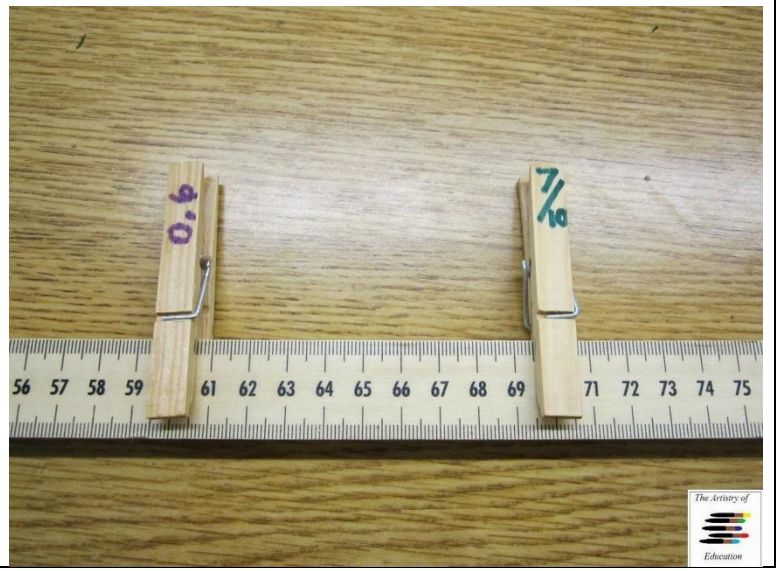
Distributive Property (Division)	Primary Grades: 3-8 States that dividing a sum by a number is the same as dividing each addend by the number and then adding the quotients	
Line Plot	Primary Grades: 2-8 A representation of data (like a graph) using a number line and x's to represent the data	

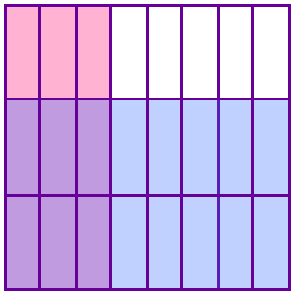
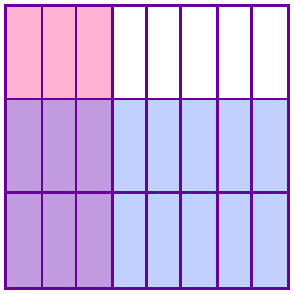
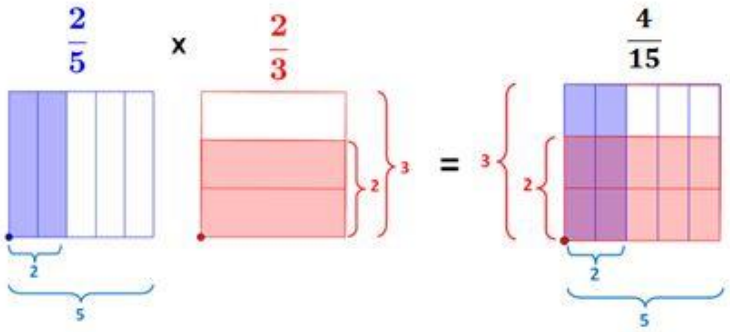
Partial Products	Primary Grades: 4-6 Used for multiplication A method of multiplying in which the ones, tens, hundreds and so on are multiplied separately and then the products are added together	
Partial Quotients	Primary Grades: 4-6 Used for division A method of dividing in which multiples of the divisor are subtracted from the dividend and then the quotients are added together	

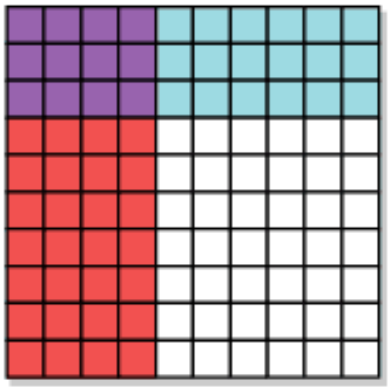
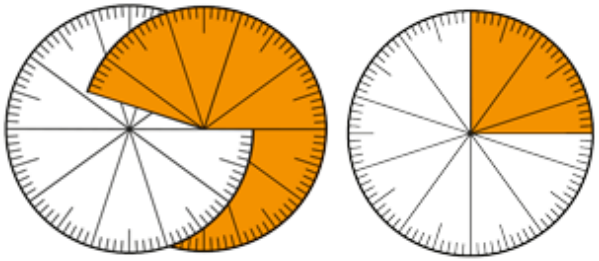
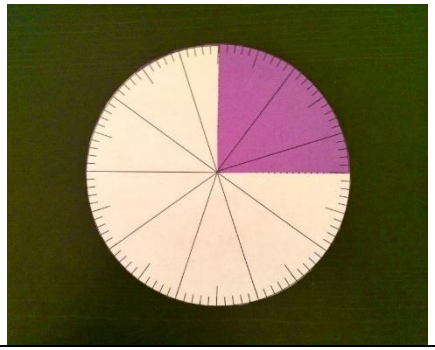
Area Model for Fractions	Primary Grades: 2-8 A way to pictorially represent fractions	 
Three-Dimensional Volume Model of Fractions (relative volume)	Primary Grades: 2-8 The advantage of this model is that the 'whole' is obvious. This is also a disadvantage as the amount of material inside is hidden from view. Not all young students have a good idea of volume and so may be observing something else (e.g. surface area).	 

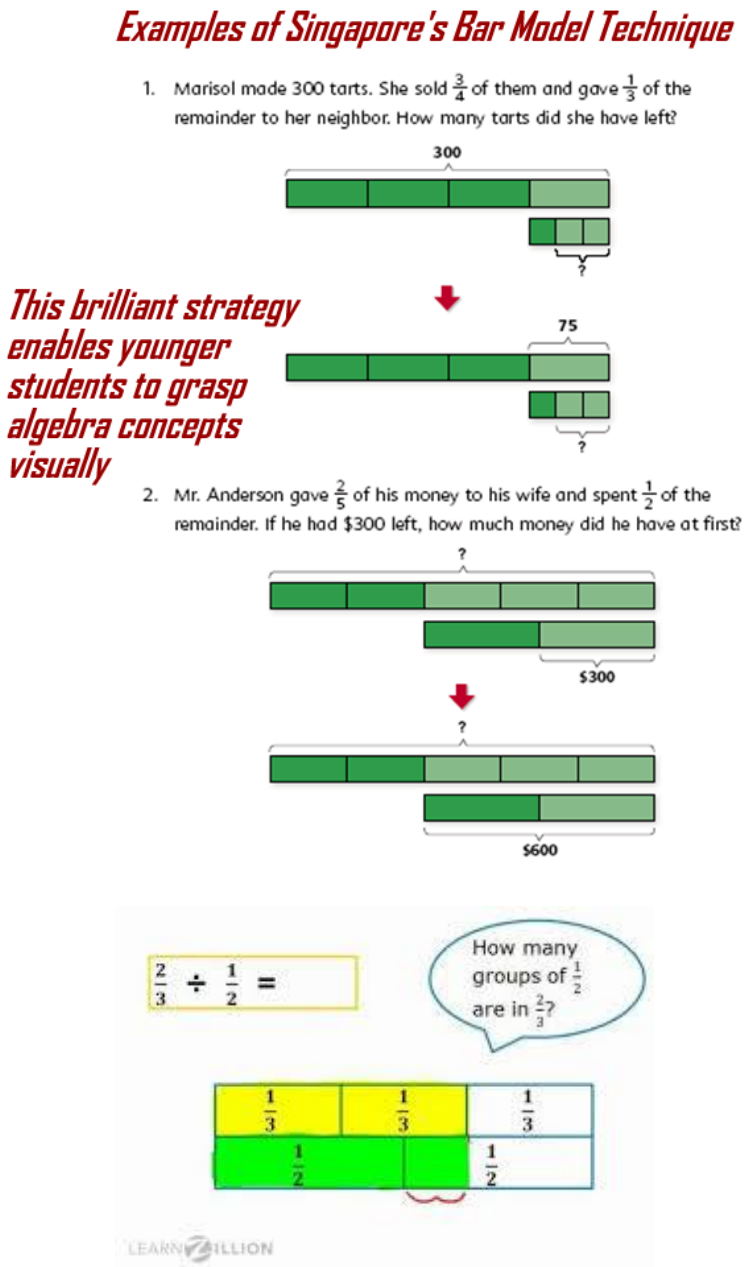
One-dimensional linear model of fractions (relative length)	Primary Grades 2-5 A linear model gives a good feel for relative size of the numbers and may be shown with fraction strips. Need to be clear about what length is the 'whole'.	
Discrete Model of Fractions – Subset / Set model (relative number)	Primary Grades 2-5 Need to be clear about what set of objects is to be regarded as the 'whole' and then identify fractions of this whole. Can be modeled easily with counters. This can reinforce the idea that fractions are two numbers rather than one (2 of the 5 counters are black).	

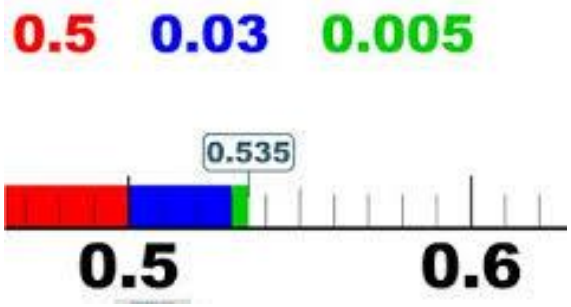
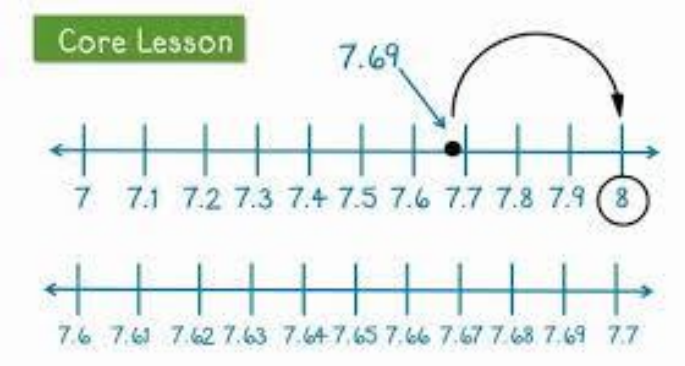
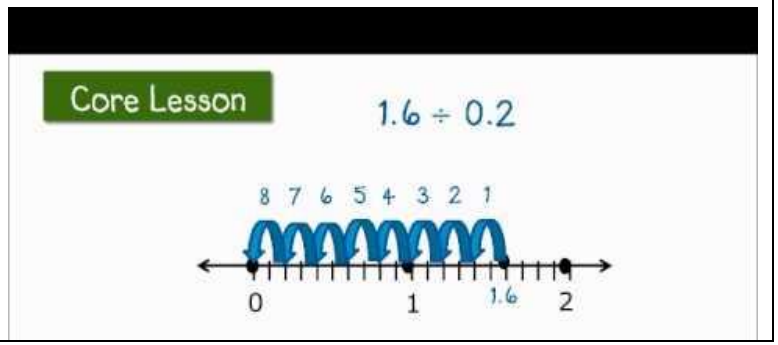
Number line Model of Fractions (position of a point on a line)	Primary Grades: 2-8 A linear model (e.g. a fraction strip) represents numbers by length. A number line is made by representing each number by a point at that distance (length) from the origin. The number line is a more sophisticated concept than a linear model, which is an important pre-requisite. This model is especially useful for operations with fractions, particularly multiplication and division.	 

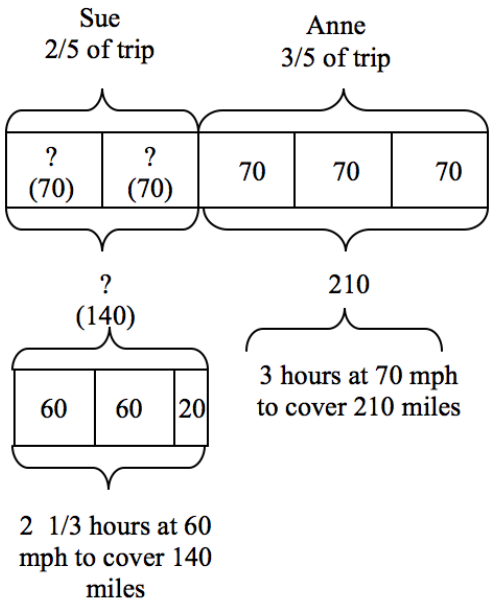
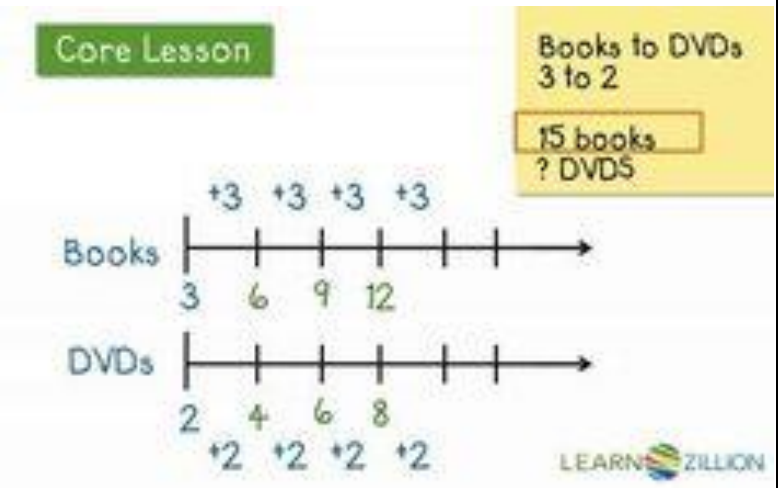
Length Model of Decimals (meter stick)	Primary Grades: 4-6 A meter stick can be used to represent the length model showing both tenths (decimeters) and hundredths (centimeters). This concrete model transfers well to semi-concrete models such as number lines drawn with 10 or 100 divisions, and help students make quantity comparisons between decimals and base ten fractions.	 

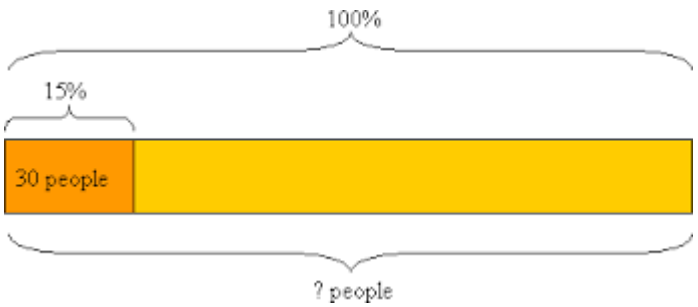
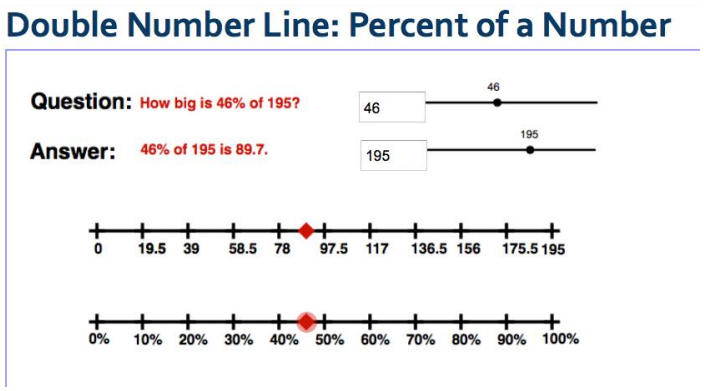
Two-dimensional Area Model of Fractions (relative area)	Primary Grades 4-8 The whole can be made obvious. Can be hard to compare amounts – the whole must be the same size. Not all young students have a good idea of area and so may be observing something else (e.g. perimeter). Can be used to model operations with fractions, especially useful for understanding multiplication and division of fractions.	 $\frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$  $\frac{3}{8} \cdot \frac{2}{3} = \frac{3 \cdot 2}{8 \cdot 3}$ 

Area Model of Multiplying Decimals	Primary Grades: 4-8 Model the product of two decimals by finding the area of a rectangle. Estimate the area of the rectangle first. Then break the rectangle into several pieces and find the area of each piece (partial product). Add these areas together to find the whole area (product).	$0.3 \times 0.4 = 0.12$ 
Hundredths Wheel	Primary Grades 4-8 This model helps students convert between fractions, decimals, and percentages. What's great about hundredth wheels is they provide students with a great visual representation of decimal values that is easy to relate to familiar visual representations of fractions, like 0.25 and 1/4.	 

Bar Model / Tape Diagram of Dividing Fractions	Primary Grades 4-8 To model division with fractions, we start with an area we're looking for, and we find one of the missing factors that makes up that area.	

Number Line Model for Decimals	Primary Grades: 4-8 To represent a decimal on a number line, divide each segment of the number line into ten equal parts. Especially useful for rounding decimals and representing multiplication and division of decimals.	  

Bar Model / Tape Diagram for Ratios and Rates	Primary Grades 6-8 The Bar Model method requires students to draw diagrams in the form of rectangular bars to represent known and unknown quantities, as well as the relationships between these quantities.	
Double Number Line Diagrams for Ratios	Primary Grades 6-8 Double number line diagrams are best used when the quantities have different units. Double number line diagrams can help us to see that there are many, even infinitely many, pairs of numbers in the same ratio—including those with rational number entries.	

Tape Diagram for Percentages	Primary Grades: 6-8 The tape diagram requires students to draw diagrams in the form of rectangular bars to represent known and unknown quantities, as well as the relationships between these quantities.	
Double Number Line Models for Percentages	Primary Grades: 6-8 A double number line is a number line with a scale on top and a different scale on the bottom so that you can organize and compare items that change regularly according to a rule or pattern. Double number lines can be used for fractions, decimals, and percentages.	

Mathematical Modeling Websites:

- **National Library of Virtual Manipulatives** http://nlvm.usu.edu/en/nav/vlibrary.html
  - Interactive manipulatives in all areas: number & operations, algebra, geometry, measurement, data analysis and probability. Organized by grade level PK-12.

- **Dreambox.com interactive activities** http://www.dreambox.com/teachertools
  - Great site. Broken down by grade level (K-5), lots of fun interactive activities to reinforce important skills.
  - Site uses such tools as: Open Number lines, Rekenreks, Ten Frames, Number Bonds (Part Part Whole), Area Models (Partial Products Multiplication), Partial Quotients (Division Strategy) and Visual Fraction Models.

- **Problem Solving using Tape Diagrams (Bar Models)** http://www.mathplayground.com/thinkingblocks.html
  - Great practice setting up word problems using rectangular bars to model the problem and decide how to go about solving it. Great resource. Check it out.

- **Interactive Rekenreks** http://www.ronblond.com/MathGlossary/Division01/REKENREK/index.html
  - Useful for building early number sense, helps students to work off of the anchor numbers 5 and 10. Great visual for seeing addition and subtraction problems.

- **Interactive Fraction Number line** http://www.mathisfun.com/numbers/fraction-number-line.html
  - Not sure how many fourths is equal to six eighths? Here's the interactive tool for you.

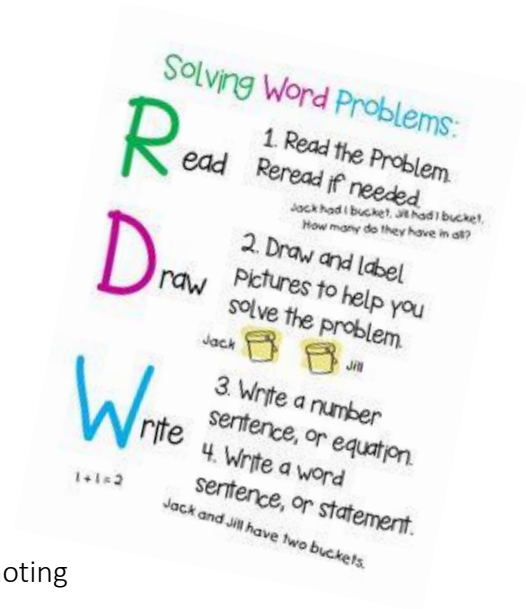
- **Interactive Place Value Disks** http://exchange.smarttech.com/details.html?id=8cd88c08-522c-4984-bf99-c01f9dd49ba3
  - If you access the website you'll see a tab to download the place value disks. When you click on it you will be asked to sign up. It's a free site and a great resource. Enjoy.

- **Interactive Algebra Tiles** http://illuminations.nctm.org/Activity.aspx?id=3482
  - Use tiles to represent variables and constants, learn how to represent and solve algebra problem. Solve equations, substitute in variable expressions, and expand and factor. Flip tiles, remove zero pairs, copy and arrange, and make your way toward a better understanding of algebra.

- **Interactive Fraction Models** http://illuminations.nctm.org/Activity.aspx?id=3519
  - Explore different representations for fractions including improper fractions, mixed numbers, decimals, and percentages. Additionally, there are length, area, region, and set models. Adjust numerators and denominators to see how they alter the representations and models. Use the table to keep track of interesting fractions.

- **Interactive Pan Balance – Expressions** http://illuminations.nctm.org/Activity.aspx?id=3529
  - This interactive pan balance allows numeric or algebraic expressions to be entered and compared. You can "weigh" the expressions you want to compare by entering them on either side of the balance. Using this interactive tool, you can practice arithmetic and algebraic skills, and investigate the important concept of equivalence.

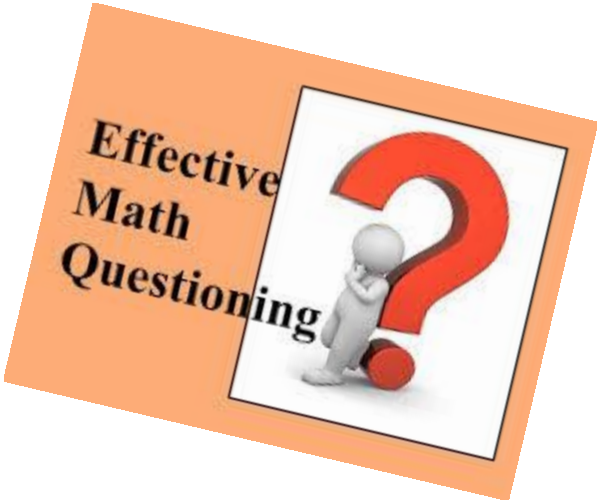
Read, Draw, Write (RDW) Problem-Solving Process



- Mathematicians and teachers suggest a simple process of problem solving applicable to all grades:
  1) Read.
  2) Draw and Label.
  3) Write a number sentence (equation).
  4) Write a word sentence (statement).

- The Read, Draw, Write (RDW) process for solving word problems is modeled to students by teachers, promoting perseverance in reasoning through problems.
  - Students may ask themselves these questions to guide them through the problem solving process:
    - "What do I see?"
    - "Can I draw something?"
    - "What can I draw?"
    - "What can I learn from my drawing?"
    - "What conclusions can I make from my drawing?"
  - After drawing, students write a statement responding to the question. The more students participate in reasoning through problems with a systematic approach, the more they internalize those behaviors and thought processes.

Guiding Questions for Math



In helping children learn, one goal is to assist children in figuring out as much as they can for themselves (constructing meaning). You can help by asking questions that guide, without telling what to do. Good questions and good listening will help children make sense of mathematics, build self-confidence, and encourage mathematical thinking and communication. A good question opens up a problem and supports different ways of thinking about it. Here are some questions you might try; notice that none of them can be answered with a simple "yes" or "no."

Getting Started

What do you need to find out?
What do you need to know?
How can you get that information?
Where can you begin?
What terms do you understand or not understand?
Have you solved similar problems that would help?
What similar examples can you find?

While Working

How can you organize the information?
Can you make a drawing (model) to explain your thinking?
Are there other possibilities?
What would happen if…….?
Can you describe an approach (strategy) you can use to solve this?
What do you need to do next?
Do you see any patterns or relationships that will help solve this?
How does this relate to…..?
Can you make a prediction?
What did you…..?
What assumptions are you making?

Reflecting about the solution

How do you know your solution (conclusion) is reasonable?

How did you arrive at your answer?

How can you convince me your answer makes sense?

What did you try that did not work?

Has the question been answered?

Can your explanation be made clearer?

Responding (helping your children clarify and extend their thinking)

Tell me more about this.

Can you explain it in a different way?

Is there another possibility or strategy that would work?

Help me understand this part . . .